

The Feynman-Vernon Functional of Electromagnetic Field Influence on Spinor Field

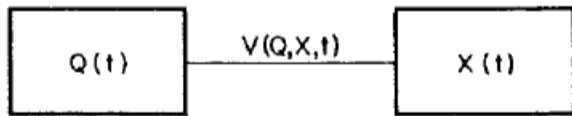
Alexander Biryukov, Mark Shleenkov

Samara National Research University

*Physics of fundamental interactions
dedicated of 50th anniversary of Bakhan Neutrino Observatory
Nalchik, Kabardino-Balkaria, June 6–8, 2017*

The Feynman-Vernon Influence Functional approach

R.P. Feynman, F.L. Vernon, Jr. **The Theory of a General Quantum System Interacting with a Linear Dissipative System**, *Annals of Physics* **24**, 118–173.



General quantum systems Q and X coupled by a potential $V(Q, X, t)$.

«... It is shown that the effect of the external systems in such a formalism [paths integral formalism] can always be included in a general class of functionals (influence functionals) of the coordinates of the system only...»

QED Lagrangian

$$\mathcal{L}(x) = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - ej^\mu(x)A_\mu(x) \quad (1)$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \quad (2)$$

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \quad (3)$$

$$\hat{\psi}(\mathbf{x}, t) = \sum_{\mathbf{p}, \sigma=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{p}}^{(f)}}} \left(\hat{b}_{\mathbf{p}\sigma} u_{\sigma}(p) e^{ipx} + \hat{c}_{\mathbf{p}\sigma}^{\dagger} u_{\sigma}(-p) e^{-ipx} \right), \quad (4)$$

$$\hat{\bar{\psi}}(\mathbf{x}, t) = \sum_{\mathbf{p}, \sigma=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{p}}^{(f)}}} \left(\hat{b}_{\mathbf{p}\sigma}^{\dagger} \bar{u}_{\sigma}(p) e^{-ipx} + \hat{c}_{\mathbf{p}\sigma} \bar{u}_{\sigma}(-p) e^{ipx} \right), \quad (5)$$

$$\hat{j}_{\mu}(\mathbf{x}, t) = \hat{\bar{\psi}}(\mathbf{x}, t) \gamma_{\mu} \hat{\psi}(\mathbf{x}, t) \quad (6)$$

$$\hat{A}^{\mu}(\mathbf{x}, t) = \sum_{\mathbf{k}, \lambda=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}^{(b)}}} \varepsilon_{\lambda}^{\mu} \left(\hat{a}_{\mathbf{k}\lambda} e^{ikx} + \hat{a}_{\mathbf{k}\lambda}^{\dagger} e^{-ikx} \right) \quad (7)$$

Second quantization

$$\begin{aligned}
 \hat{H}_{full} = & \sum_{\mathbf{p}, \sigma=1,2} \omega_{\mathbf{p}}^{(f)} \left(\hat{b}_{\mathbf{p}\sigma}^\dagger \hat{b}_{\mathbf{p}\sigma} + \hat{c}_{\mathbf{p}\sigma}^\dagger \hat{c}_{\mathbf{p}\sigma} \right) + \sum_{\mathbf{k}, \lambda=1,2} \omega_{\mathbf{k}}^{(b)} \hat{a}_{\mathbf{k}\lambda}^\dagger \hat{a}_{\mathbf{k}\lambda} + \\
 & + e \sum_{\mathbf{k}, \lambda=1,2} \frac{i}{\sqrt{2\omega_{\mathbf{k}}^{(b)} V}} \left(\varepsilon_{\lambda}^{\mu} \hat{j}_{\mu}^{+}(\mathbf{k}, t) \hat{a}_{\mathbf{k}\lambda} + \varepsilon_{\lambda}^{*\mu} \hat{j}_{\mu}^{-}(\mathbf{k}, t) \hat{a}_{\mathbf{k}\lambda}^\dagger \right) \quad (8)
 \end{aligned}$$

where

$$\hat{j}_{\mu}^{+}(\mathbf{k}, t) = \int \hat{j}_{\mu}(\mathbf{x}, t) e^{i\mathbf{k}\mathbf{x}} d\mathbf{x}, \quad \hat{j}_{\mu}^{-}(\mathbf{k}, t) = \int \hat{j}_{\mu}(\mathbf{x}, t) e^{-i\mathbf{k}\mathbf{x}} d\mathbf{x} \quad (9)$$

Evolution equation for statistical operator $\hat{\rho}(t_f)$

$$\hat{\rho}(t_f) = \hat{U}(t_f, t_{in})\hat{\rho}(t_{in})\hat{U}^\dagger(t_f, t_{in}) \quad (10)$$

where $\hat{\rho}(t_{in})$ is statistical operator, describing initial state at moment t_{in} ,
 $\hat{U}(t_f, t_{in})$ — evolution operator.

$$\hat{U}(t_f, t_{in}) = \hat{T} \exp\left[-\frac{i}{\hbar} \int_{t_{in}}^{t_f} \hat{H}_{full}(\tau) d\tau\right]. \quad (11)$$

where \hat{H}_{full} :

$$\hat{H}_{full} = \hat{H}_{sys} + \hat{H}_{field} + \hat{H}_{int} \quad (12)$$

Coherent states for electromagnetic field

$$\hat{a}_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle = \alpha_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle, \quad \langle\alpha_{\mathbf{k}\lambda}|\hat{a}_{\mathbf{k}\lambda}^\dagger = \langle\alpha_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}^*, \quad (13)$$

where $\alpha_{\mathbf{k}\lambda}$ — complex value, which describe states \mathbf{k} mode of quantum electromagnetic field. These states ($|\alpha\rangle$) are non-orthogonal:

$$\langle\alpha'_{\mathbf{k}'\lambda'}|\alpha_{\mathbf{k}\lambda}\rangle = \delta_{\mathbf{k}'\mathbf{k}}\delta_{\lambda'\lambda} \exp\left\{-\frac{1}{2}(|\alpha'_{\mathbf{k}'\lambda'}|^2 + |\alpha_{\mathbf{k}\lambda}|^2 - 2\alpha'_{\mathbf{k}'\lambda'}^*\alpha_{\mathbf{k}\lambda})\right\}. \quad (14)$$

There is resolution of the identity operator:

$$\int |\alpha_{\mathbf{k}\lambda}\rangle\langle\alpha_{\mathbf{k}\lambda}| \frac{d^2\alpha_{\mathbf{k}\lambda}}{\pi} = \hat{1}. \quad (15)$$

Grassman states for Dirac field

$$\hat{b}_{\mathbf{p},\sigma}|\theta_{\mathbf{p},\sigma}\rangle = \theta_{\mathbf{p},\sigma}|\theta_{\mathbf{p},\sigma}\rangle, \quad \langle\bar{\theta}_{\mathbf{p},\sigma}|\hat{b}_{\mathbf{p},\sigma}^\dagger = \langle\bar{\theta}_{\mathbf{p},\sigma}|\bar{\theta}_{\mathbf{p},\sigma}, \quad (16)$$

where $\theta_{\mathbf{p},\sigma}$ — grassman variable. These states ($|\theta\rangle$) are non-orthogonal:

$$\langle\bar{\theta}'_{\mathbf{p}'\sigma'}|\theta_{\mathbf{p}\sigma}\rangle = \delta_{\mathbf{p}'\mathbf{p}}\delta_{\sigma'\sigma} \exp\left\{-\frac{1}{2}\left(\bar{\theta}'_{\mathbf{p}'\sigma'}\theta'_{\mathbf{p}'\sigma'} + \bar{\theta}_{\mathbf{p}\sigma}\theta_{\mathbf{p}\sigma} - 2\bar{\theta}'_{\mathbf{p}'\sigma'}\theta_{\mathbf{p}\sigma}\right)\right\}. \quad (17)$$

There is resolution of the identity operator:

$$\int |\theta_{\mathbf{p}\sigma}\rangle\langle\bar{\theta}_{\mathbf{p}\sigma}| \frac{d^2\theta_{\mathbf{p}\sigma}}{\pi} = \hat{1}. \quad (18)$$

Grassman variables properties

For two grassman variables θ and η

$$\theta\eta + \eta\theta = 0 \quad \text{or} \quad \theta\eta = -\eta\theta \quad \text{so} \quad \theta^2 = 0 \quad (19)$$

Then

$$\int d\theta f(\theta) = \int d\theta (A + B\theta) = B. \quad (20)$$

and

$$\int d\theta^* d\theta e^{-\theta^* b \theta} = b \quad (21)$$

and

$$\left(\prod_i \int d\theta_i^* d\theta_i \right) \theta_k \theta_l^* e^{-\theta_i^* B_{ij} \theta_j} = \frac{\det B}{B_{kl}} \quad (22)$$

Evolution equation for density matrix in holomorphic representation $|\theta_{p\sigma}, \alpha_{k\lambda}\rangle = |\theta_{p\sigma}\rangle \otimes |\alpha_{k\lambda}\rangle$

The density matrix:

$$\rho(\alpha_f^*, \bar{\theta}_f, \alpha'_f, \theta'_f; t_f) = \langle \bar{\theta}_f, \alpha_f | \hat{\rho}(t_f) | \theta'_f, \alpha'_f \rangle \quad (23)$$

The kernel of evolution operator:

$$U(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) = \langle \bar{\theta}_f, \alpha_f | \hat{U}(t_f, t_{in}) | \theta_{in}, \alpha_{in} \rangle \quad (24)$$

The evolution equation:

$$\begin{aligned} \rho(\alpha_f^*, \bar{\theta}_f, \alpha'_f, \theta'_f; t_f) &= \int \frac{d^2\alpha'_{in}}{\pi} \frac{d^2\theta'_{in}}{\pi} \frac{d^2\alpha_{in}}{\pi} \frac{d^2\theta_{in}}{\pi} \times \\ &\times U(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) \rho(\alpha_{in}^*, \bar{\theta}_{in}, \alpha'_{in}, \theta'_{in}; t_{in}) U^*(\alpha'_f, \theta'_f, t_f | \alpha_{in}^*, \bar{\theta}'_{in}; t_{in}) \end{aligned} \quad (25)$$

The kernel of evolution operator

$$\begin{aligned}
 U(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_f, \theta_f, t_{in}) &= \int \mathcal{D}\alpha^*(\tau) \mathcal{D}\alpha(\tau) \mathcal{D}\bar{\theta}(\tau) \mathcal{D}\theta(\tau) \times \\
 &\times \exp \left\{ i S_{full} [\alpha^*(\tau), \alpha(\tau), \bar{\theta}(\tau), \theta(\tau)] \right\}, \quad (26)
 \end{aligned}$$

where action

$$\begin{aligned}
 &S_{full} [\alpha^*(\tau), \alpha(\tau), \bar{\theta}(\tau), \theta(\tau)] = \\
 &= S_f [\bar{\theta}(\tau), \theta(\tau)] + S_b [\alpha^*(\tau), \alpha(\tau)] + S_{int} [\alpha^*(\tau), \alpha(\tau), \bar{\theta}(\tau), \theta(\tau)]. \quad (27)
 \end{aligned}$$

Action of fermionic field:

$$S_f [\bar{\theta}(\tau), \theta(\tau)] = \int_{t_{in}}^{t_f} \left(\frac{\dot{\bar{\theta}}(\tau)\theta(\tau) - \bar{\theta}(\tau)\dot{\theta}(\tau)}{2i} - \omega^{(f)}\bar{\theta}(\tau)\theta(\tau) \right) d\tau \quad (28)$$

Action of bosonic field:

$$S_b [\alpha^*(\tau), \alpha(\tau)] = \int_{t_{in}}^{t_f} \left(\frac{\dot{\alpha}^*(\tau)\alpha(\tau) - \alpha^*(\tau)\dot{\alpha}(\tau)}{2i} - \omega^{(b)}\alpha^*(\tau)\alpha(\tau) \right) d\tau \quad (29)$$

Action of interaction part:

$$S_{int} [\alpha^*(\tau), \alpha(\tau), \bar{\theta}(\tau), \theta(\tau)] = \int_{t_{in}}^{t_f} e j^\mu(\bar{\theta}(\tau), \theta(\tau)) (\varepsilon_\mu^* \alpha^*(\tau) + \varepsilon_\mu \alpha(\tau)) d\tau; \quad (30)$$

Evolution of density matrix in paths integral formulation

We have

$$\begin{aligned}
 \rho(\alpha_f^*, \bar{\theta}_f, \alpha'_f, \theta'_f; t_f) = & \int \frac{d^2\alpha'_{in}}{\pi} \frac{d^2\theta'_{in}}{\pi} \frac{d^2\alpha_{in}}{\pi} \frac{d^2\theta_{in}}{\pi} \rho(\alpha_{in}^*, \bar{\theta}_{in}, \alpha'_{in}, \theta'_{in}; t_{in}) \times \\
 & \times \mathcal{D}\alpha^*(\tau) \mathcal{D}\alpha(\tau) \mathcal{D}\bar{\theta}(\tau) \mathcal{D}\theta(\tau) \mathcal{D}\alpha'^*(\tau) \mathcal{D}\alpha'(\tau) \mathcal{D}\bar{\theta}'(\tau) \mathcal{D}\theta'(\tau) \times \\
 & \times \exp \left\{ i \left(S_{full} [\alpha^*(\tau), \alpha(\tau), \bar{\theta}(\tau), \theta(\tau)] - S_{full} [\alpha'^*(\tau), \alpha'(\tau), \bar{\theta}'(\tau), \theta'(\tau)] \right) \right\}, \quad (31)
 \end{aligned}$$

Fermionic density matrix and influence functional

$$\begin{aligned} \rho(\bar{\theta}_f, \theta'_f; t_f) &= Sp_{\alpha_f = \alpha'_f} \rho(\alpha_f^*, \bar{\theta}_f, \theta'_f, \alpha'_f; t_f) = \int d\theta'_f d\theta_f \mathfrak{D}\bar{\theta}(\tau) \mathfrak{D}\theta(\tau) \mathfrak{D}\bar{\theta}'(\tau) \mathfrak{D}\theta'(\tau) d\theta'_{in} d\theta_{in} \times \\ &\times \exp \left\{ i \left(S_f[\bar{\theta}(\tau), \theta(\tau)] - S_f[\bar{\theta}'(\tau), \theta'(\tau)] \right) \right\} F[\theta(\tau), \theta'(\tau)] \end{aligned} \quad (32)$$

where $F[\bar{\theta}(\tau), \theta'(\tau)]$ is influence functional of electromagnetic field on fermionic subsystems.

$$\begin{aligned} F[\bar{\theta}(\tau), \theta'(\tau)] &= Sp_{\alpha_f = \alpha'_f} \int \mathfrak{D}\alpha^*(\tau) \mathfrak{D}\alpha(\tau) \mathfrak{D}\alpha'^*(\tau) \mathfrak{D}\alpha'(\tau) \frac{d^2\alpha_{in}}{\pi} \frac{d^2\alpha'_{in}}{\pi} \rho(\alpha_{in}^*, \bar{\theta}_{in}, \alpha'_{in}, \theta'_{in}; t_{in}) \times \\ &\times \exp \left\{ i \left(S_b[\alpha^*(\tau), \alpha(\tau)] + S_{int}[\alpha^*(\tau), \alpha(\tau), \bar{\theta}(\tau), \theta(\tau)] - S_b[\alpha'^*(\tau), \alpha'(\tau)] - S_{int}[\alpha'^*(\tau), \alpha'(\tau), \bar{\theta}'(\tau), \theta'(\tau)] \right) \right\} \end{aligned} \quad (33)$$

In many cases, we can choose at initial moment t_{in}

$$\rho(\alpha_{in}^*, \bar{\theta}_{in}, \alpha'_{in}, \theta'_{in}; t_{in}) = \rho_f(\bar{\theta}_{in}, \theta'_{in}; t_{in}) \times \rho_b(\alpha_{in}^*, \alpha'_{in}; t_{in}) \quad (34)$$

Influence functional of electromagnetic field

$$F[\bar{\theta}(\tau), \theta'(\tau)] = \int Sp_{\alpha_f = \alpha'_f} \frac{d^2 \alpha_{in}}{\pi} \frac{d^2 \alpha'_{in}}{\pi} \times \\ \times U_{infl}(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) \rho(\alpha_{in}^*, \bar{\theta}_{in}, \alpha'_{in}, \theta'_{in}; t_{in}) U_{infl}^*(\alpha'_f, \theta'_f, t_f | \alpha_{in}^*, \bar{\theta}'_{in}; t_{in}) \quad (35)$$

where $U_{infl}(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in})$ is electromagnetic field transition amplitude from initial state $|\alpha_{in}\rangle$ to final state $|\alpha_f^*\rangle$ inducing by external source j :

$$U_{infl}(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) = \int \mathfrak{D}\alpha^*(\tau) \mathfrak{D}\alpha(\tau) \exp \{i S_{infl}[\alpha^*(\tau), \alpha(\tau), x(\tau)]\} \quad (36)$$

where $S_{infl}[\alpha^*(\tau), \alpha(\tau), \bar{\theta}(\tau), \theta(\tau)] = S_b[\alpha^*(\tau), \alpha(\tau)] + S_{int}[\alpha^*(\tau), \alpha(\tau), \bar{\theta}(\tau), \theta(\tau)]$. In general, influence functional (26) describes the influence (action) of electromagnetic field on fermionic field.

Functional integration over electromagnetic field paths

$$\begin{aligned}
 U_{infl}(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) = \exp \left\{ e^{-i\omega(t_f - t_{in})} \alpha_f^* \alpha_{in} - e^2 \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \varepsilon^\mu j_\mu^+(\tau) \varepsilon^{*\nu} j_\nu^-(\tau') e^{i\omega(\tau - \tau')} d\tau d\tau' - \right. \\
 \left. - i\alpha_{in} e \int_{t_{in}}^{t_f} \varepsilon^\mu j_\mu^+(\tau) e^{-i\omega(\tau - t_{in})} d\tau - i\alpha_f^* e \int_{t_{in}}^{t_f} \varepsilon^{*\mu} j_\mu^-(\tau) e^{-i\omega(t_f - \tau)} d\tau \right\}
 \end{aligned} \tag{37}$$

For multimode field and two polarizations without interaction between modes

$$U_{infl} = \prod_{\mathbf{k}, \lambda} U_{infl}^{(\mathbf{k}, \lambda)} \tag{38}$$

$$\begin{aligned}
& U_{infl}(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) = \\
& = \prod_{\mathbf{k}, \lambda=1,2} \exp \left\{ e^{-i\omega_{\mathbf{k}}(t_f - t_{in})} \alpha_{\mathbf{k}\lambda}^{(f)*} \alpha_{\mathbf{k}\lambda}^{(in)} - \frac{e^2}{2\omega_{\mathbf{k}}V} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \varepsilon_{\lambda}^{\mu} j_{\mu}^{+}(\mathbf{k}, \tau) \varepsilon_{\lambda}^{*\nu} j_{\nu}^{-}(\mathbf{k}, \tau') e^{i\omega_{\mathbf{k}}(\tau - \tau')} d\tau d\tau' - \right. \\
& \left. - i\alpha_{\mathbf{k}\lambda}^{(in)} \frac{e}{\sqrt{2\omega_{\mathbf{k}}V}} \int_{t_{in}}^{t_f} \varepsilon_{\lambda}^{\mu} j_{\mu}^{+}(\mathbf{k}, \tau) e^{-i\omega_{\mathbf{k}}(\tau - t_{in})} d\tau - i\alpha_{\mathbf{k}\lambda}^{(f)*} \frac{e}{\sqrt{2\omega_{\mathbf{k}}V}} \int_{t_{in}}^{t_f} \varepsilon_{\lambda}^{*\mu} j_{\mu}^{-}(\mathbf{k}, \tau) e^{-i\omega_{\mathbf{k}}(t_f - \tau)} d\tau \right\} = \\
& = \exp \left\{ \sum_{\mathbf{k}, \lambda} \left(e^{-i\omega_{\mathbf{k}}(t_f - t_{in})} \alpha_{\mathbf{k}\lambda}^{(f)*} \alpha_{\mathbf{k}\lambda}^{(in)} - \frac{e^2}{2\omega_{\mathbf{k}}V} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \varepsilon_{\lambda}^{\mu} j_{\mu}^{+}(\mathbf{k}, \tau) \varepsilon_{\lambda}^{*\nu} j_{\nu}^{-}(\mathbf{k}, \tau') e^{i\omega_{\mathbf{k}}(\tau - \tau')} d\tau d\tau' - \right. \right. \\
& \left. \left. - i\alpha_{\mathbf{k}\lambda}^{(in)} \frac{e}{\sqrt{2\omega_{\mathbf{k}}V}} \int_{t_{in}}^{t_f} \varepsilon_{\lambda}^{\mu} j_{\mu}^{+}(\mathbf{k}, \tau) e^{-i\omega_{\mathbf{k}}(\tau - t_{in})} d\tau - i\alpha_{\mathbf{k}\lambda}^{(f)*} \frac{e}{\sqrt{2\omega_{\mathbf{k}}V}} \int_{t_{in}}^{t_f} \varepsilon_{\lambda}^{*\mu} j_{\mu}^{-}(\mathbf{k}, \tau) e^{-i\omega_{\mathbf{k}}(t_f - \tau)} d\tau \right) \right\} \quad (39)
\end{aligned}$$

Vacuum influence functional

For the case when initial and final states of electromagnetic field are vacuum:

$$\phi_{in}(\alpha_{in}) = \langle \alpha_{in} | 0 \rangle = \exp \left\{ -\frac{1}{2} |\alpha_{in}|^2 \right\}, \quad \phi_f^*(\alpha_f) = \langle 0 | \alpha_f \rangle = \exp \left\{ -\frac{1}{2} |\alpha_f|^2 \right\}. \quad (40)$$

We define influence functional of electromagnetic vacuum

$$\begin{aligned} F_{\langle vac|vac \rangle}[\bar{\theta}(\tau), \theta'(\tau)] &= \int \frac{d^2 \alpha_f}{\pi} \frac{d^2 \alpha'_f}{\pi} \frac{d^2 \alpha_{in}}{\pi} \frac{d^2 \alpha'_{in}}{\pi} \rho_f(\bar{\theta}_{in}, \theta'_{in}; t_{in}) \times \\ &\quad \times \phi_f^*(\alpha_f) U_{infl}(\alpha_f^*, \bar{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) \phi_{in}(\alpha_{in}) \phi_{in}^*(\alpha'_{in}) U_{infl}^*(\alpha'_f, \theta'_f, t_f | \alpha'_{in}, \bar{\theta}'_{in}; t_{in}) \phi_f^*(\alpha'_f) = \\ &= \exp \left\{ -\sum_{\mathbf{k}, \lambda} \frac{e^2}{2\omega_{\mathbf{k}} V} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \left(\varepsilon_{\lambda}^{\mu} j_{\mu}^{+}(\mathbf{k}, \tau) \varepsilon_{\lambda}^{*\nu} j_{\nu}^{-}(\mathbf{k}, \tau') e^{i\omega_{\mathbf{k}}(\tau - \tau')} d\tau d\tau' + \varepsilon_{\lambda}^{\mu} j_{\mu}^{'+}(\mathbf{k}, \tau) \varepsilon_{\lambda}^{*\nu} j_{\nu}^{-}(\mathbf{k}, \tau') e^{-i\omega_{\mathbf{k}}(\tau - \tau')} d\tau d\tau' \right) \right\} \end{aligned} \quad (41)$$

From sum over \mathbf{k} to integral: $\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int d\mathbf{k}$

$$\begin{aligned}
 & - \sum_{\mathbf{k}} \frac{e^2}{2\omega_{\mathbf{k}} V} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{* \nu} \right] j_{\mu}^{+}(\mathbf{k}, \tau) j_{\nu}^{-}(\mathbf{k}, \tau') e^{i\omega_{\mathbf{k}}(\tau - \tau')} d\tau d\tau' = \\
 & = - \frac{e^2}{(2\pi)^3} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \int \frac{1}{2\omega_{\mathbf{k}}} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{* \nu} \right] j_{\mu}^{+}(\mathbf{k}, \tau) j_{\nu}^{-}(\mathbf{k}, \tau') e^{i\omega(\tau - \tau')} d\mathbf{x} d\mathbf{x}' d\mathbf{k} d\tau d\tau' = \\
 & = - \frac{e^2}{(2\pi)^3} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \int \frac{1}{2\omega_{\mathbf{k}}} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{* \nu} \right] j_{\mu}(\mathbf{x}, \tau) j_{\nu}(\mathbf{x}', \tau') e^{-i\mathbf{k}(\mathbf{x} - \mathbf{x}')} e^{i\omega(\tau - \tau')} d\mathbf{x} d\mathbf{x}' d\mathbf{k} d\tau d\tau' = \\
 & = - \frac{e^2}{4\pi i} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \int \underbrace{\left[\frac{1}{(2\pi)^3} \int \frac{2\pi i d\mathbf{k}}{\omega_{\mathbf{k}}} \left(\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{* \nu} \right) e^{-i\mathbf{k}(\mathbf{x} - \mathbf{x}')} e^{i\omega(\tau - \tau')} d\mathbf{k} \right]}_{D^{\mu\nu}(\mathbf{x} - \mathbf{x}', \tau - \tau')} j_{\mu}(\mathbf{x}, \tau) j_{\nu}(\mathbf{x}', \tau') d\mathbf{x} d\mathbf{x}' d\tau d\tau'
 \end{aligned}$$

where $D^{\mu\nu}(\mathbf{x} - \mathbf{x}', \tau - \tau')$ is photon propagator ¹.

$$F_{\langle vac|vac \rangle}[\bar{\theta}(\tau), \theta'(\tau)] = \exp \left\{ -\frac{e^2}{4\pi\imath} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \int j_{\mu}(\mathbf{x}, \tau) D^{\mu\nu}(\mathbf{x} - \mathbf{x}', \tau - \tau') j_{\nu}(\mathbf{x}', \tau') d\mathbf{x} d\mathbf{x}' d\tau d\tau' - \right. \\ \left. - \frac{e^2}{4\pi\imath} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \int j'_{\mu}(\mathbf{x}, \tau) D^{\mu\nu}(\mathbf{x} - \mathbf{x}', \tau - \tau') j'_{\nu}(\mathbf{x}', \tau') d\mathbf{x} d\mathbf{x}' d\tau d\tau' \right\} \quad (42)$$

For $t_f \rightarrow \infty$, $t_{in} \rightarrow -\infty$ we have relativistic invariant influence functional of electromagnetic vacuum:

$$F_{\langle vac|vac \rangle}[\bar{\theta}(\tau), \theta'(\tau)] = \exp \left\{ -\frac{e^2}{4\pi\imath} \int \int (j_{\mu}(x) D^{\mu\nu}(x - x') j_{\nu}(x') + j'_{\mu}(x) D^{\mu\nu}(x - x') j'_{\nu}(x')) d^4x d^4x' \right\} \quad (43)$$

So we have effective Lagrangian

$$\mathcal{L} = \bar{\psi}(x)(i\gamma_{\mu}\partial^{\mu} - m)\psi(x) + \frac{e^2}{4\pi}j_{\mu}(x) \int D^{\mu\nu}(x - x')j_{\nu}(x')dx' \quad (44)$$

The Euler–Lagrange equation for this model is

$$(i\gamma_{\mu}\partial^{\mu} - m)\psi(x) + \frac{e^2}{4\pi} \left[\int D^{\mu\nu}(x - x')j_{\nu}(x')dx' \right] \gamma_{\mu}\psi(x) = 0 \quad (45)$$

where

$$D^{\mu\nu}(x - x') = \frac{1}{(2\pi)^3} \int \frac{2\pi i d\mathbf{k}}{\omega_{\mathbf{k}}} \left(\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{*\nu} \right) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{i\omega(\tau-\tau')} \quad (46)$$

and

$$j_{\nu}(x') = \bar{\psi}(x')\gamma_{\nu}\psi(x') \quad (47)$$

We note that obtained equation is non-linear. We present equation (45) in two forms:

$$(i\gamma_{\mu}\partial^{\mu} - \mathcal{M})\psi(x) = 0 \quad (48)$$

where

$$\mathcal{M} = m - \frac{e^2}{4\pi} \left[\int D^{\mu\nu}(x - x') j_{\nu}(x') dx' \right] \gamma_{\mu} \quad (49)$$

Thanks for your attention!