РОСАТОМ

Р Ф Я Ц
ВНИИЭФ

ГОСУДАРСТВЕННАЯ КОРПОРАЦИЯ ПО АТОМНОЙ ЭНЕРГИИ «РОСАТОМ»

Atomic systems with degenerate bound states of fermions in the Schwarzschild, Reissner-Nordström fields – candidates for the role of the dark matter particles

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1. Introduction

the Coulomb potential $-\frac{Ze^2}{r}$

electron

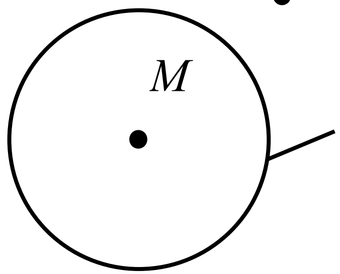
$(-e, m)$

nucleus

$(Ze, M_я)$

the Schwarzschild field

(m)



$$r_0 = \frac{2GM}{c^2}$$

$$\alpha_{em} = Z\alpha_{fs} = \frac{Ze^2}{\hbar c}$$

$$Z = 1; \ \alpha_{fs} = \frac{1}{137}$$

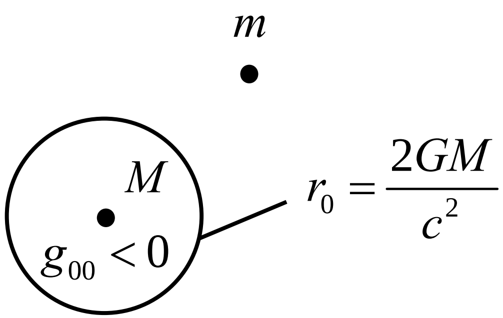
$$\alpha = \frac{GMm}{\hbar c} = \frac{Mm}{M_p^2} = \frac{r_0}{2l_c}$$

$$M_p = 2,2 \cdot 10^{-5} \ \text{g} = 1,2 \cdot 10^{19} \ Gev$$

$$M = M_{\odot} = 2 \cdot 10^{30} \ \text{kg}; \ m = m_e; \ \alpha = 4 \cdot 10^{15}$$

$$\rho = \frac{r}{l_c}; \quad \frac{r_0}{l_c} = 2\alpha$$

the Schwarzschild field



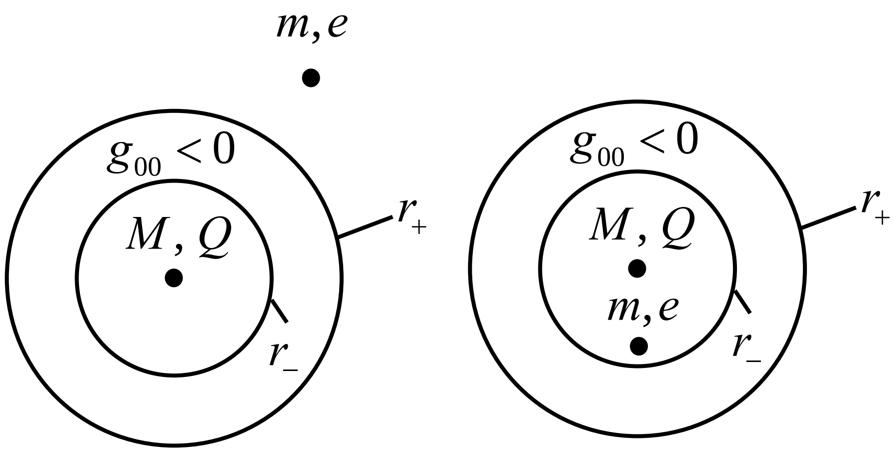
$$\alpha = \frac{GMm}{\hbar c} = \frac{Mm}{M_p^2} = \frac{r_0}{2l_c}$$

$$M_p = 2,2 \cdot 10^{-5}\ \text{g} = 1,2 \cdot 10^{19}\ Gev$$

$$M = M_{\odot} = 2 \cdot 10^{30}\ \text{kg};\ m = m_e;\ \alpha = 4 \cdot 10^{15}$$

$$\rho = \frac{r}{l_c};\quad \frac{r_0}{l_c} = 2\alpha$$

the Reissner-Nordström field



$$r_\pm = \frac{r_0}{2} \pm \sqrt{\frac{r_0^2}{4} - r_Q^2};\quad r_Q = \frac{\sqrt{G}|Q|}{c^2}$$

$$\rho = \frac{r}{l_c};\quad \alpha_Q = \frac{r_Q}{l_c} = \frac{\sqrt{G}QM}{\hbar c} = \frac{\sqrt{\alpha_{fs}}}{M_p} m \frac{Q}{|e|}$$

$$\alpha_{em} = \frac{eQ}{\hbar c} = \alpha_{fs} \frac{Q}{e}$$

$$\rho_\pm = \alpha^2 \pm \sqrt{\alpha^2 - \alpha_Q^2};\ \alpha^2 > \alpha_Q^2$$

2. Is it possible to implement the existence condition of stationary bound states of fermions in the external gravitational fields?

Reissner-Nordström and Schwarzschild metrics

$$ds^2 = f_{R-N}dt^2 - \frac{dr^2}{f_{R-N}} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

$$g_{00}^{R-N} = f_{R-N} = \left(1 - \frac{r_0}{r} + \frac{r_Q^2}{r^2}\right) = \left(1 - \frac{2\alpha}{\rho} + \frac{\alpha_Q^2}{\rho^2}\right)$$

As $r_Q = 0$ $\rightarrow$ **the Schwarzschild metric is**

$$g_{00}^{s} = f_s = 1 - \frac{r_0}{r} = 1 - \frac{2\alpha}{\rho}$$

The Dirac equation in the Hamiltonian form

$$i\frac{\partial \Psi_\eta}{\partial t} = H_\eta \Psi_\eta$$

WHAT DO WE HAVE?

1. Self-adjoint Hamiltonian

$$H_\eta = H_\eta^+ = \sqrt{f_{R-N}}\, m\gamma^{\underline{0}} - i\gamma^{\underline{0}}\gamma^{\underline{1}}\left(f_{R-N}\frac{\partial}{\partial r} + \frac{1}{r} - \frac{r_0}{2r^2}\right) -$$

$$-i\sqrt{f_{R-N}}\,\frac{1}{r}\left[\gamma^{\underline{0}}\gamma^{\underline{2}}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\mathrm{ctg}\,\theta\right) + \gamma^{\underline{0}}\gamma^{\underline{3}}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right] + \frac{eQ}{r}.$$

2. Possibility of separation of variables

$$\Psi_\eta = \begin{pmatrix} F(r)\xi(\theta) \\ -iG(r)\sigma^3\xi(\theta) \end{pmatrix} e^{-iEt} e^{im_\varphi \varphi}$$

$$\xi(\theta) = \begin{pmatrix} {}_{-1/2}Y_{jm_\varphi}(\theta) \\ {}_{1/2}Y_{jm_\varphi}(\theta) \end{pmatrix} = (-1)^{m_\varphi + 1/2}\sqrt{\frac{1}{4\pi}\frac{(j-m_\varphi)!}{(j+m_\varphi)!}} \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \left(\kappa - m_\varphi + \frac{1}{2}\right) P_l^{m_\varphi - 1/2}(\theta) \\ P_l^{m_\varphi + 1/2}(\theta) \end{pmatrix}$$

3. Current density of the Dirac particles

$$j^0 = \Psi_\eta^+ \Psi_\eta = \left(F(\rho)F^*(\rho) + G(\rho)G^*(\rho)\right)\xi^+(\theta)\xi(\theta),$$

$$j^\rho = \Psi_\eta^+ f_{R-N}\gamma^0\gamma^3\Psi_\eta = -if_{R-N}\left(F^*(\rho)G(\rho) - F(\rho)G^*(\rho)\right)\xi^+(\theta)\xi(\theta),$$

$$j^\theta = \Psi_\eta^+ \frac{f_{R-N}^{1/2}}{\rho}\gamma^0\gamma^1\Psi_\eta = -\frac{f_{R-N}^{1/2}}{\rho}\left(F^*(\rho)G(\rho) + F(\rho)G^*(\rho)\right)\xi^+(\theta)\sigma^2\xi(\theta),$$

$$j^\varphi = \Psi_\eta^+ \frac{f_{R-N}^{1/2}}{\rho\sin\theta}\gamma^0\gamma^2\Psi_\eta = \frac{f_{R-N}^{1/2}}{\rho\sin\theta}\left(F^*(\rho)G(\rho) + F(\rho)G^*(\rho)\right)\xi^+(\theta)\sigma^1\xi(\theta).$$

3.1 **Generally speaking, for the complex radial functions, the current density j^ρ can be non-zero. In this case the Hamiltonian is non-Hermitian $(\Phi, H_\eta\Psi) \neq (H_\eta\Phi, \Psi)$. One can exist only quasi-stationary states of fermions decaying with time.**

3.2 **For the real radial functions $(F^* = F, G^* = G)$, the density of the radial current is equal to zero in the entire domain of wave functions and the Dirac Hamiltonian is Hermitian.**

Current density j^θ is equal to zero for both complex and real $F(\rho)$ and $G(\rho)$ since $\xi^+(\theta)\sigma^2\xi(\theta) = 0$. On the contrary, current density j^φ differs from zero for any functions $F(\rho)$ and $G(\rho)$.

4. **<u>It is important</u> to restrict the consideration to real radial wave functions only** $F(\rho), G(\rho)$**.**

5. **The system of equations for the real radial wave functions has the form**

$$f_{R-N}\frac{dF(\rho)}{d\rho}+\left(\frac{1+\kappa\sqrt{f_{R-N}}}{\rho}-\frac{\alpha}{\rho^2}\right)F(\rho)-\left(\varepsilon-\frac{\alpha_{em}}{\rho}+\sqrt{f_{R-N}}\right)G(\rho)=0,$$

$$f_{R-N}\frac{dG(\rho)}{d\rho}+\left(\frac{1-\kappa\sqrt{f_{R-N}}}{\rho}-\frac{\alpha}{\rho^2}\right)G(\rho)+\left(\varepsilon-\frac{\alpha_{em}}{\rho}-\sqrt{f_{R-N}}\right)F(\rho)=0.$$

The system of equations is real if $f_{R-N}\ge 0$**, i.e. the domain of definition for the functions** $F(\rho), G(\rho)$ **is the intervals** $\rho\in[\rho_+,\infty)$**,** $\rho\in(0,\rho_-]$**.**

For the Schwarzschild field, the condition $f_{R-N}\ge 0$ **leads to the domain of definition** $\rho\in[2\alpha,\infty)$**.**

_6._ The asymptotics of solutions near the event horizons

_6.1_

$$F\big|_{\rho\to\rho_+} = \frac{A}{\sqrt{\rho-\rho_+}}\sin\left(\frac{\rho_+^2}{\rho_+-\rho_-}\left(\varepsilon-\frac{\alpha_{em}}{\rho_+}\right)\ln\left(\rho-\rho_+\right)+\varphi_+\right),$$

$$G\big|_{\rho\to\rho_+} = \frac{A}{\sqrt{\rho-\rho_+}}\cos\left(\frac{\rho_+^2}{\rho_+-\rho_-}\left(\varepsilon-\frac{\alpha_{em}}{\rho_+}\right)\ln\left(\rho-\rho_+\right)+\varphi_+\right).$$

_6.2_

$$F\big|_{\rho\to\rho_-} = -\frac{B}{\sqrt{\rho_--\rho}}\sin\left(\frac{\rho_-^2}{\rho_+-\rho_-}\left(\varepsilon-\frac{\alpha_{em}}{\rho_-}\right)\ln\left(\rho_--\rho\right)+\varphi_-\right),$$

$$G\big|_{\rho\to\rho_-} = \frac{B}{\sqrt{\rho_--\rho}}\cos\left(\frac{\rho_-^2}{\rho_+-\rho_-}\left(\varepsilon-\frac{\alpha_{em}}{\rho_-}\right)\ln\left(\rho_--\rho\right)+\varphi_-\right).$$

_6.3_ For the Schwarzschild field

$$F\big|_{\rho\to 2\alpha} = \frac{A}{\sqrt{\rho-2\alpha}}\sin\left(2\alpha\varepsilon\ln\left(\rho-2\alpha\right)+\varphi\right),\quad G = \frac{A}{\sqrt{\rho-2\alpha}}\cos\left(2\alpha\varepsilon\ln\left(\rho-2\alpha\right)+\varphi\right).$$

Negative moment:

- the function $F(\rho), G(\rho)$ are square-nonintegrable in the neighborhood of the event horizon (for example, normalization integral $N=\int_{\rho_+}^{\infty}\left(F(\rho)^2+G(\rho)^2\right)\rho^2 d\rho$ is logarithmically diverged);

- for $\varepsilon \neq \alpha_{em}/\rho_+$, $\varepsilon \neq \alpha_{em}/\rho_-$ (the Reissner-Nordström field), $\varepsilon \neq 0$ (the Schwarzschild field), the presented asymptotics indicate an implementation of a “fall of the particle to the event horizons”.

Positive moment:

- for $\varepsilon = \alpha_{em} / \rho_+$, $\varepsilon = \alpha_{em} / \rho_-$ **(the Reissner-Nordström field);**

$\varepsilon = 0$ **(the Schwarzschild field) the asymototics indicate an absence of the mode of a "fall" of the fermions to the event horizons.**

Now we need to solve a problem of square-nonintegrability of the wave function for the existence of stationary bound states of fermions.

7. The Schrödinger-type equation with effective potential

Let us transform the system of the Dirac equations for the radial functions $F(\rho), G(\rho)$ to the Schrödinger-type relativistic equations for the function $\psi_F(\rho)$ proportional to $F(\rho)$ and for the function $\psi_G(\rho)$ proportional to $G(\rho)$.

$$\psi_F(\rho) = F(\rho)\exp\left(\frac{1}{2}\int^{\rho} A_F(\rho')\,d\rho'\right), \quad \psi_G(\rho) = G(\rho)\exp\left(\frac{1}{2}\int^{\rho} A_G(\rho')\,d\rho'\right),$$

where $A_F(\rho) = -\frac{1}{B}\frac{dB}{d\rho} - A - D, \quad A_G(\rho) = -\frac{1}{C}\frac{dC}{d\rho} - A - D$ **and**

$$A(\rho) = -\frac{1}{f_{R-N}}\left(\frac{1+\kappa\sqrt{f_{R-N}}}{\rho} - \frac{\alpha}{\rho^2}\right), \quad B(\rho) = \frac{1}{f_{R-N}}\left(\varepsilon + \sqrt{f_{R-N}}\right),$$

$$C(\rho) = -\frac{1}{f_{R-N}}\left(\varepsilon - \sqrt{f_{R-N}}\right), \quad D(\rho) = -\frac{1}{f_{R-N}}\left(\frac{1-\kappa\sqrt{f_{R-N}}}{\rho} - \frac{\alpha}{\rho^2}\right).$$

The equations for $\psi_F(\rho)$ and $\psi_G(\rho)$ have the form of the Schrödinger equation

$$\frac{d^2\psi_F(\rho)}{d\rho^2}+2\left(E_{Schr}-U_{eff}^F(\rho)\right)\psi_F(\rho)=0,$$

$$\frac{d^2\psi_G(\rho)}{d\rho^2}+2\left(E_{Schr}-U_{eff}^G(\rho)\right)\psi_G(\rho)=0,$$

where

$$E_{Schr}=\frac{1}{2}\left(\varepsilon^2-1\right),$$

$$U_{eff}^F(\rho)=E_{Schr}+\frac{3}{8}\frac{1}{B^2}\left(\frac{dB}{d\rho}\right)^2-\frac{1}{4}\frac{1}{B}\frac{d^2B}{d\rho^2}+\frac{1}{4}\frac{d}{d\rho}(A-D)-$$
$$-\frac{1}{4}\frac{(A-D)}{B}\frac{dB}{d\rho}+\frac{1}{8}(A-D)^2+\frac{1}{2}BC.$$

$$U_{eff}^G(\rho)=E_{Schr}+\frac{3}{8}\frac{1}{C^2}\left(\frac{dC}{d\rho}\right)^2-\frac{1}{4}\frac{1}{C}\frac{d^2C}{d\rho^2}-\frac{1}{4}\frac{d}{d\rho}(A-D)+$$
$$+\frac{1}{4}\frac{(A-D)}{C}\frac{dC}{d\rho}+\frac{1}{8}(A-D)^2+\frac{1}{2}BC.$$

8. Square-integrability $\psi_F(\rho),\ \psi_G(\rho)$

It is important that during the transition to the Schrödinger -type equation, the radial wave functions $\psi_F(\rho),\ \psi_G(\rho)$ become square-integrable in all domains of definition $\rho\in[\rho_+,\infty),\ \rho\in(0,\rho_-],\ \rho\in[2\alpha,\infty)$.

$$\psi_F\left(\varepsilon=\frac{\alpha_{em}}{\rho_+}\right)\Bigg|_{\rho\to\rho_+}=C_1\left(\rho-\rho_+\right)^{1/4},$$

$$\psi_F\left(\varepsilon=\frac{\alpha_{em}}{\rho_-}\right)\Bigg|_{\rho\to\rho_-}=C_2\left(\rho_--\rho\right)^{1/4},$$

$$\psi_F\left(\varepsilon=0\right)\Big|_{\rho\to 2\alpha}=C_3\left(\rho-2\alpha\right)^{1/4}.$$

The wave functions equal to zero on event horizons $\rho_+,\rho_-,2\alpha$.

3. Degenerate stationary bound states of half-part particles in the external Schwarzschild and Reissner-Nordström fields

the Schawazschild field

$$\varepsilon_S = 0$$

the Reissner-Nordström field

$$\rho_+ = \alpha + \sqrt{\alpha^2 - \alpha_Q^2}; \quad \rho_- = \alpha - \sqrt{\alpha^2 - \alpha_Q^2},$$

$$\varepsilon_{R-N} = \alpha_{em} / \rho_+, \quad \rho \in [\rho_+, \infty],$$

$$\varepsilon_{R-N} = \alpha_{em} / \rho_-, \quad \rho \in (0, \rho_+].$$

Everywhere, the interval $-1 \le \varepsilon \le 1$ is allowed energy range of a particle in the bound state.

For each of the degenerate solution, there exist rather different for various j, l square-integrable wave eigenfunctions being the solutions to the Schrödinger-type equation with the effective potencials.

3.1 Stationary bound states of fermion particles in the external Kerr and Kerr-Newman fields

the Kerr field

$$\rho_+ = \alpha + \sqrt{\alpha^2 - \alpha_a^2}; \;\; \rho_- = \alpha - \sqrt{\alpha^2 - \alpha_a^2},$$

$$\varepsilon_K = \frac{m_\varphi \alpha_a}{\alpha_a^2 + \rho_+^2}, \;\; \rho \in [\rho_+, \infty).$$

$$\varepsilon_K = \frac{m_\varphi \alpha_a}{\alpha_a^2 + \rho_-^2}, \;\; \rho \in (0, \rho_-].$$

the Kerr-Newman field

$$\rho_+ = \alpha + \sqrt{\alpha^2 - \alpha_a^2 - \alpha_Q^2}; \;\; \rho_- = \alpha - \sqrt{\alpha^2 - \alpha_a^2 - \alpha_Q^2},$$

$$\varepsilon_{K-N} = \frac{m_\varphi \alpha_a + \alpha_{em}\rho_+}{\rho_+^2 + \alpha_a^2}, \;\; \rho \in [\rho_+, \infty).$$

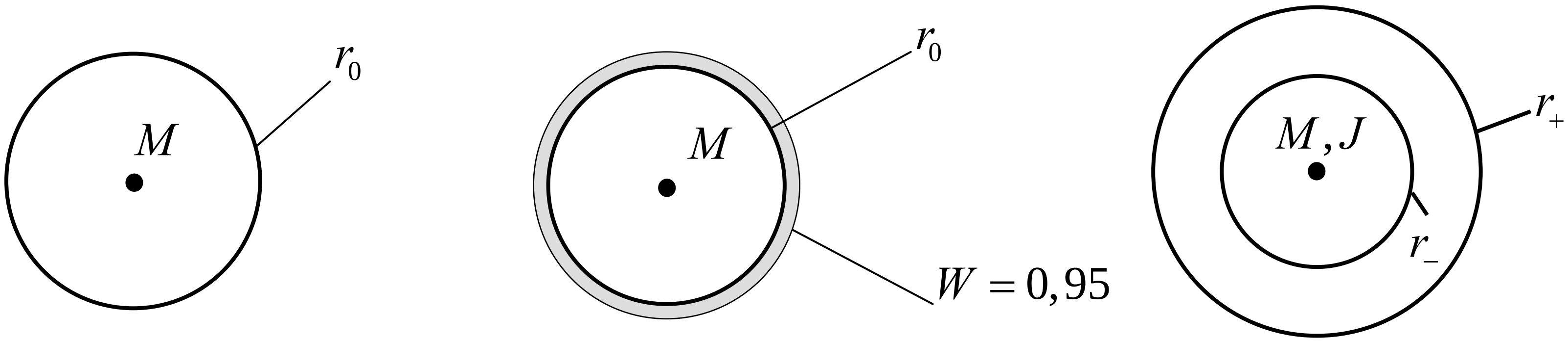
$$\varepsilon_{K-N} = \frac{m_\varphi \alpha_a + \alpha_{em}\rho_-}{\alpha_a^2 + \rho_-^2}, \;\; \rho \in (0, \rho_-].$$

4. Schwarzschild, Reissner-Nordström collapsars with and without degenerate stationary bound states of fermions – candidates for the role of the dark matter particles

Degenerate stationary bound state of half-spin particles in the field of collapsars can be good candidates for the role of dark matter particles.

4.1 **Let us consider solution for the Schwarzschild field** $\varepsilon_S = 0$**. If one neglects the gravitational interaction of uncharged Dirac particles, then for the Schwarzschild collapsars with mass** M **an atomic system of bound half-spin particles with** $\varepsilon_S = 0$ **is possible. Degenerate states with different** j, l **values should be filled with taking account of the Pauli principle. The analogy, for instance, is an atom of hydrogen with the degenerate state with respect to the orbital moment** l**.**

The atomic system, the Schwarzschild collapsar with uncharged Dirac particles with $\varepsilon_S = 0$, interacts with other objects only gravitationally. Due to the absence of quantum transitions between the states with different j, l, a system like this neither emits nor absorbs light or other kinds of radiation. This system can be detected only through gravitational interaction. Masses of such systems should be chosen from the condition of the best agreement with the Standard cosmological model.

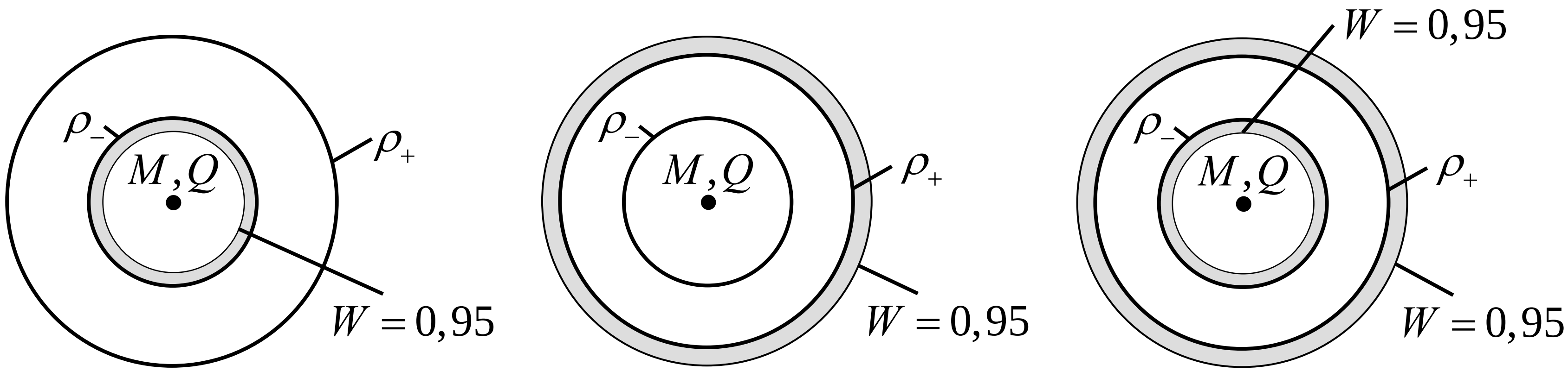


4.2 Solutions for the Reissner-Nordström field (RN)

Let us consider solution $\varepsilon = \alpha_{em} / \rho_-$. If at RN collapsar formation, an atomic system with bound fermions localizing near the inner neighbourhood of event horizon ρ_- appears and charge source of RN field is compensated by the total charge of the bound fermions than, for the environment, such atomic system interacts with other objects only gravitationally. Due to the absence of quantum transitions between the states with different κ (or j,l), a system like this neither emits nor absorbs light or other kinds of radiation. This system can be detected only through gravitational interaction.

The second atomic system can be a system of bound fermions in RN field with energy $\varepsilon = \alpha_{em} / \rho_+$. In this case, the fermions are, with the overwhelming probability, near the outer neighbourhood ρ_+ and, if the RN field source is compensated by the total charge of the bound fermions, such atomic system interacts with other objects only gravitationally. As in the first case, the atomic system neither emits nor absorbs light or other kinds of radiation. The RN field source in this atomic system can be detected only through "knocking-out" a part of fermions out of their orbits by means of external action.

The other atomic systems with energy of the bound fermions in part with $\varepsilon = \alpha_{em}/\rho_+$, in part with $\varepsilon = \alpha_{em}/\rho_-$ can be considered as candidates for the role of the dark matter particles. Masses of such systems should be chosen from the condition of the best agreement with the Standard cosmological model.



The situation is not changed qualitatively

if there is a rotation (the Kerr, Kerr-Newman field)!!!

Remark 1

Many authors, A.G.Riess at al, D.Kovetz, P.Frampton and al. draw a conclusion that the most viable candidate for the constituent dark matter is the Primordial Intermediate Mass Black Hole (PIMBH) with mass in the range

$$M_{PIMBH} \leq 10^5 M_{\odot}$$

The existence of the accretion of matter and emission of X-rays by accreted matter is typical for the dark holes.

The difficulty for the models with PIMBH, as a dark matter, is the lack of any observed departure of the CMB spectrum from predicted black-body curve or any CMB anisotropy.

But, for the collapsars considered in this presentation, there exist no possibility of particles penetration through the outer event horizon and emission of X-rays by accreted matter will be significant smaller.

Remark 2

If the interaction of fermions with the Schwarzschild collapsars is described by the real radial functions then

1. **the radial Dirac current** $j_S^{\rho} = 0$**,**
2. **singularity of the Schrödinger-type equation**

$$U_S^{eff}\Big|_{\rho \to 2\alpha} = -\frac{3}{32}\frac{1}{\left(\rho - 2\alpha\right)^2}.$$

Both equalities are remained at transition to the Painleve-Gullstrand coordinates although the coordinate singularity of the Schwarzschild metrics formally is disappeared!!!

5. Conclusions

The consideration of the solution of the Schrödinger-type equation with the effective potentials in the quantum mechanics of motion of fermions in the classical Schwarzschild and Reissner-Nordström fields leads to the following results:

1. Availability of the event horizons $2\alpha, \rho_+, \rho_-$, there are regular solutions with energies $\varepsilon = 0$ (the Schwarzschild field), $\varepsilon = \alpha_{em}/\rho_+$, $\varepsilon = \alpha_{em}/\rho_-$ (the RN field). These solutions represent the degenerate stationary bound states of charged and uncharged fermions with the square-integrable wave functions and domains of definition $\rho \in [2\alpha, \infty), \rho \in [\rho_+, \infty)$, $\rho \in (0, \rho_-)$. The wave functions depend insignificantly on j, l and become zero on the event horizons. With the overwhelming probability, the fermions in the bound states are near the event horizons. The maxima of the probability densities are away from the event horizons within the range from several fractions to unit of the Compton wavelength of the fermions.

2. For the extreme RN field with the single event horizon $\rho_+ = \rho_- = \alpha$, the absence of the stationary bound states of half-spin particles was proved.

3. For the naked singularity of the RN field $\left(\alpha^2 < \alpha_Q^2\right)$, at the certain values of physical parameters, the analysis of the effective potentials and direct numerical solutions of the Schrödinger-type equation showed the existence of the stationary bound states of the charged fermions of the RN field. There are the bound states also for the electrically uncharged fermions; this states are implemented by means of the gravitational interaction only.

4. For any quantum - mechanical half-spin particle, irrespective of availability and sign of the electrical charge, the RN naked singularity is separated by the infinitely high positive potential barrier $\sim \frac{3}{8}\frac{1}{r^2}$. The availability of a repulsive barrier shielding the singularity not be seen as a threat to the cosmic censorship.

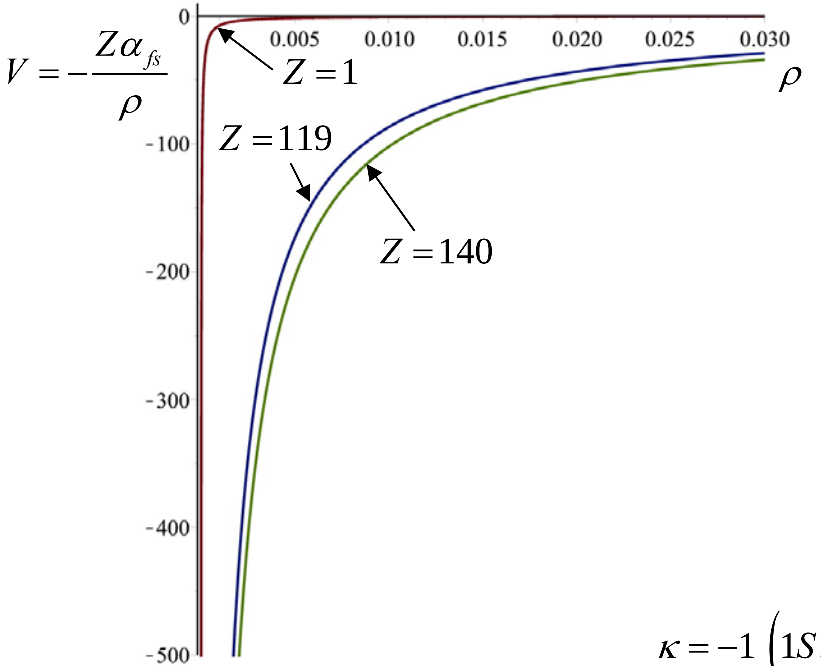
5. The electrically neutral atomic systems with a definite number of fermions being in degenerate bound states with $\varepsilon = 0$ (the Schwarzschild field), $\varepsilon = \alpha_{em}/\rho_+$, $\varepsilon = \alpha_{em}/\rho_-$ (the RN field), can be considered as particles of the dark matter in the Standard cosmological model. Such atomic systems of this a type neither absorb nor emit light or other kinds of radiation and interact with the environment only gravitationally.

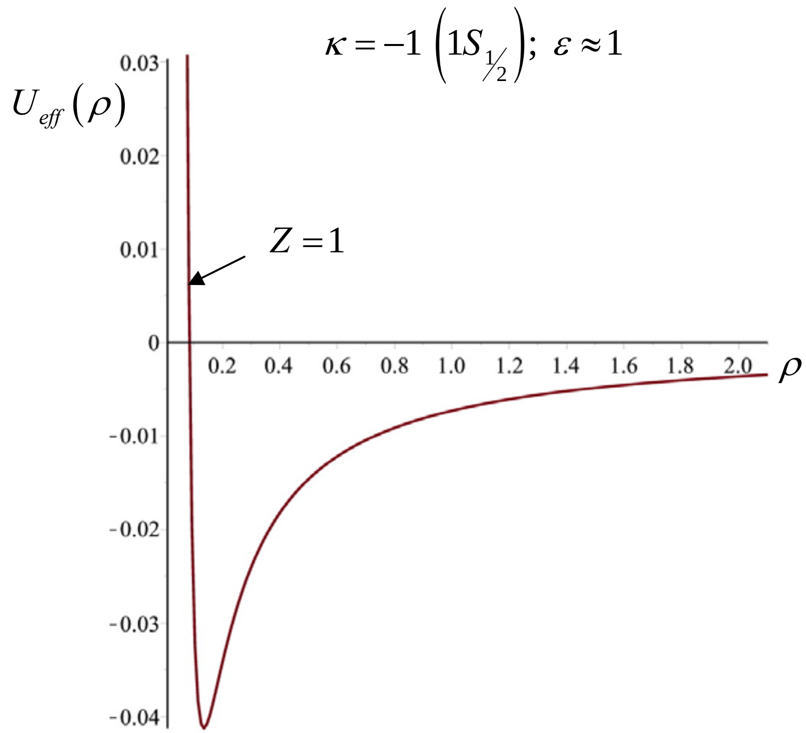
6. In spite of removal of degeneracy in respect to the magnetic quantum number m_φ, the atomic systems with fermions being in the stationary bound states with $\varepsilon = \frac{m_\varphi \alpha_a}{\alpha_a^2 + \rho_+^2}$, $\varepsilon = \frac{m_\varphi \alpha_a}{\alpha_a^2 + \rho_-^2}$ (the Kerr field); $\varepsilon = \frac{m_\varphi \alpha_a + \alpha_{em}\rho_+}{\alpha_a^2 + \rho_+^2}$, $\varepsilon = \frac{m_\varphi \alpha_a + \alpha_{em}\rho_-}{\alpha_a^2 + \rho_-^2}$ (the Kerr-Newman field) can be also considered as particles of the dark matter at the certain conditions.

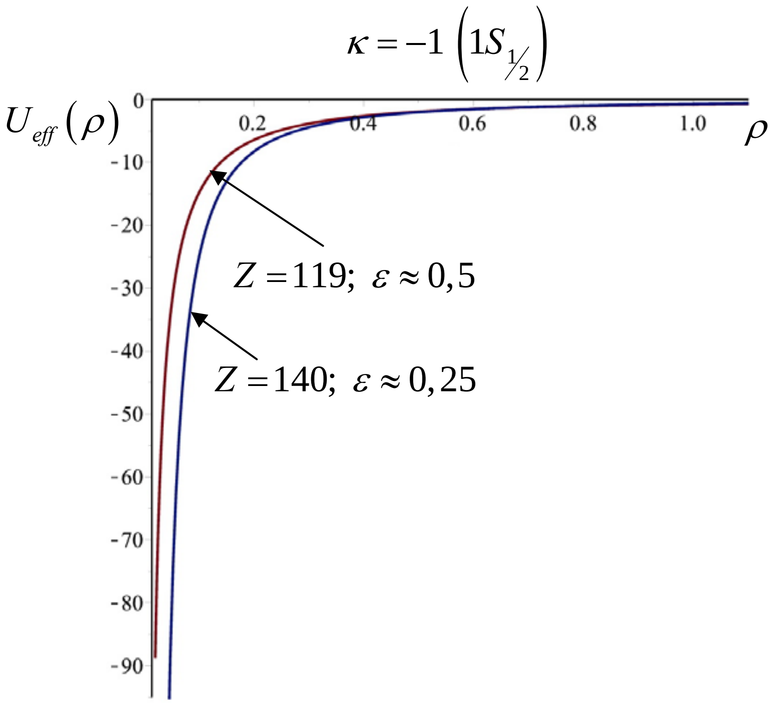
The regular solutions for the degenerate bound states with fermion energies $\varepsilon = 0$ (the Schwarzschild metric) and $\varepsilon = \alpha_{em} / \rho_+$, $\varepsilon = \alpha_{em} / \rho_-$ (the Reissner-Nordström metric) were obtained with use of the Schrödinger-type equation with effective potentials. The wave function is connected with the one of the radial functions of the Dirac equation by non-unitary transformation. As a result, the wave function of the Schrödinger-type equation for the degenerate stationary states become square-integrable in the neighborhood of the event horizon ρ_+, ρ_- unlike the radial functions of the Dirac equation. The Schrödinger-type equation can be also derived by squaring of the covariant Dirac-Foke equation in the Non-Euclidean space-time with transition from bispinor to spinor wave function and appropriate non-unitary transformation.

For the flat Minkowski space, the covariant second-order equation for fermions moving in the external electromagnetic fields was proposed by P.Dirac in the thirtieth of 20th century.

Using of the Schrödinger-type equation with origin from the relativistic second-order equation extends possibility of obtaining of the regular solutions the quantum-mechanics of the half-spin particle motion in the external gravitational fields.







$$U_{eff}\Big|_{\rho\to 0}=-\frac{\left(Z\alpha_{fs}\right)^2-3/4+\left(1-\kappa^2\right)}{2\rho^2}$$

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Thank you for your attention