Exact solutions in models of cosmological inflation with Gauss-Bonnet gravity

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EINSTEIN-GAUSS-BONNET GRAVITY

Despite the fact that the inflationary scenario solves the problem of the big bang theory, for instance, the horizon and flatness problem, there are still unsolved problems such as initial singularity problem, and quantum gravity.

For the very early Universe approaching the Planck scale, we could consider Einstein gravity with some corrections as the effective theory of the ultimate quantum gravity. The simplest such correction is the Gauss-Bonnet (GB) term in the low-energy effective action of the heterotic string.

Also, the Gauss-Bonnet term arises in the second order of Lovelock gravity, which is the generalization of the Einstein gravity. In the four-dimensional space, in the case of non-minimal interaction of a scalar field with Gauss-Bonnet term, the equations of the dynamics are quite different from the standard inflation.

- **THE EXACT SOLUTIONS OF SCALAR FIELD DYNAMICS**

- **THE INFLUENCE OF EGB TERM ON THE DYNAMICS**

THE DYNAMICAL EQUATIONS

We consider the action with the Gauss-Bonnet term non-minimally coupled to a scalar field

$$S = \int_{\mathcal{M}} d^4x\sqrt{-g}\Big[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{2}\xi(\phi)R^2_{\rm GB}\Big],$$

where ϕ is an inflation field with a potential $V(\phi)$, R the Ricci scalar curvature of the spacetime $\mathcal{M}$, $R^2_{\rm GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ the Gauss-Bonnet term. The Gauss-Bonnet coupling $\xi(\phi)$ is required to be a function of a scalar field in order to give nontrivial effects on the background dynamics.

The background dynamical equations for inflation with the GB term which couples to a scalar field ϕ in a spatially flat FRW Universe in the system of units $8\pi G = c = 1$ are

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{3k}{a^2} + 12\dot{\xi}H\left(H^2 + \frac{k}{a^2}\right) \quad (1)$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 + \frac{k}{a^2} + 2\ddot{\xi}\left(H^2 + \frac{k}{a^2}\right) + 2\dot{\xi}H\left(2\dot{H} - H^2 - \frac{3k}{a^2}\right) \quad (2)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} + 12\xi_{,\phi}\left(H^2 + \frac{k}{a^2}\right)\left(\dot{H} + H^2\right) = 0, \quad (3)$$

Let us mention about one interesting interpretation of an open and closed universes as a spatially flat one filled by the perfect fluid with the relation $3k = \rho_{m0}$. In such approach, we have the inflationary stage which is driven by the scalar field in the spatially-flat Friedmann universe ($k = 0$).
If $\xi = const$, the equations (1)–(2) are reduced to the equations for standard scalar field inflation

$$3H_E^2 = \frac{1}{2}\dot{\phi}_E^2 + V_E(\phi) - \frac{3k}{a_E^2} \tag{4}$$

$$\dot{H}_E = -\frac{1}{2}\dot{\phi}_E^2 + \frac{k}{a_E^2} \tag{5}$$

where H_E, a_E, ϕ_E and V_E are Hubble parameter, scale factor, scalar field and the potential for standard inflation with Einstein gravity.

CONFORMITY BETWEEN STANDARD FRIEDMANN AND 4D EGB COSMOLOGY

The basis of a new method for constructing exact solutions of equations (1)–(2) and determining the influence of non-minimal coupling on dynamics is the functional relationship between standard and 4D EGB cosmology $H_E = f(H, \dot{\xi}, a, k)$ which is defined in such a way that, in the case of $\xi = const$ one has $H = H_E$, $a = a_E$ and equations (1)–(2) will be reduced to (4)–(5), i.e. $\phi = \phi_E$, $V = V_E$.

$$H_E = H - 2\dot{\xi}\left(H^2 + \frac{k}{a^2}\right) \qquad (6)$$

The equations (1)–(2), in this case, can be rewritten as

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) = -3H^2 + 6HH_E + \frac{3k}{a^2} \qquad (7)$$

$$\frac{1}{2}\dot{\phi}^2 = -\dot{H}_E + HH_E - H^2 + \frac{k}{a^2} \qquad (8)$$

$\xi = const$ implies $H = H_E$, $a = a_E$ than the equations (7)–(8) are reduced to (4)–(5).

NEW DYNAMICAL EQUATIONS

We can rewrite the dynamical equations in the following form

$$V(\phi) = -2H^2 + 5HH_E + \dot{H}_E + \frac{2k}{a^2} \tag{9}$$

$$\frac{1}{2}\dot{\phi}^2 = -\dot{H}_E + HH_E - H^2 + \frac{k}{a^2} \tag{10}$$

$$\dot{\xi} = \frac{H - H_E}{2\left(H^2 + \frac{k}{a^2}\right)} \tag{11}$$

Since the equations (9)–(12) contain five unknown functions, to generate the exact solutions without additional conditions it is necessary to set two of them.

The influence of ξ on the dynamics

$$H - H_E = 2\dot{\xi}\left(H^2 + \frac{k}{a^2}\right) \tag{12}$$

THE EXACT SOLUTIONS FROM GIVEN EVOLUTIONS OF SCALE FACTOR AND SCALAR FIELD

Now, we consider the cosmological model with De Sitter expansion

$$a(t) = a_0 \exp(At), \quad H = A \tag{13}$$

$$\phi(t) = Bt, \tag{14}$$

where $A(>0)$ and B are arbitrary constants.
Form equations (9)–(12) we obtain

$$H_E(t) = A + \frac{B^2}{2A} - \frac{k}{3Aa_0^2} e^{-2At} \tag{15}$$

$$\xi(t) = -\left(\frac{1}{12A^4} + \frac{B^2}{8A^4}\right) \ln\left|A^2 a_0^2 + ke^{-2At}\right| - \frac{B^2}{4A^3} t + const \tag{16}$$

$$\xi(\phi) = -\left(\frac{1}{12A^4} + \frac{B^2}{8A^4}\right) \ln\left|A^2 a_0^2 + ke^{-\frac{2A}{B}\phi}\right| - \frac{B}{4A^3}\phi + const \tag{17}$$

$$V(\phi) = 3A^2 + \frac{5}{2}B^2 + \frac{k}{a_0^2} e^{-\frac{2A}{B}\phi}, \tag{18}$$

For a spatially-flat Friedmann universe ($k = 0$) we have

$$H_E(t) = A + \frac{B^2}{2A} \tag{19}$$

$$\xi(\phi) = -\left(\frac{1}{12A^4} + \frac{B^2}{8A^4}\right) \ln A^2 a_0^2 - \frac{B}{4A^3}\phi + const \tag{20}$$

$$V(\phi) = 3A^2 + \frac{5}{2}B^2 \tag{21}$$

If $B = 0$, we have $H = H_E = A$, $\xi = const$, $V = V_E = 3A^2$ and $\phi = \phi_E = 0$.

Now, we consider a power-law scale factor evolution and a logarithmic evolution of the scalar field:

$$a(t) = a_0 t^m, \quad H = m/t \tag{22}$$
$$\phi(t) = C \ln(Bt), \tag{23}$$

where $m(>0)$, $B(>0)$ and C are arbitrary constants.
The coupling function ξ for an arbitrary m is defined from (12) in quadratures only.
Nevertheless, an explicit dependence $\xi = \xi(t)$ and, respectively, $\xi = \xi(\phi)$ can be found for the specific values of m. Explicit integration, for example, can be performed for the model with $m = 2$.
Setting $m = 2$, form equations (9)–(12) we obtain

$$H_E(t) = \frac{C^2}{6t} + \frac{4}{3t} - \frac{k}{5a_0^2 t^3} \tag{24}$$

$$\xi(t) = \left(-\frac{C^2}{96} + \frac{1}{24}\right) B^2 t^2 + \left(\frac{kC^2}{384a_0^2} + \frac{k}{480a_0^2}\right) \ln\left|4a_0^2 t^2 + k\right| + const \tag{25}$$

$$\xi(\phi) = \left(-\frac{C^2}{96} + \frac{1}{24}\right) e^{\frac{2\phi}{C}} + \left(\frac{kC^2}{384a_0^2} + \frac{k}{480a_0^2}\right) \ln\left|\frac{4a_0^2}{B^2} e^{\frac{2\phi}{C}} + k\right| + const \quad (26)$$

$$V(\phi) = \frac{B^2}{10a_0^2}\left(15C^2 a_0^2 e^{\frac{2\phi}{C}} + 40a_0^2 e^{\frac{2\phi}{C}} + 6kB^2\right) e^{-\frac{4\phi}{C}} \quad (27)$$

For a spatially-flat Friedmann universe ($k = 0$) we have the following solution:

$$H_E(t) = \frac{C^2}{6t} + \frac{4}{3t} \quad (28)$$

$$\xi(\phi) = \left(-\frac{C^2}{96} + \frac{1}{24}\right) e^{\frac{2\phi}{C}} + const \quad (29)$$

$$V(\phi) = \frac{B^2}{10}\left(15C^2 + 40\right) e^{-\frac{2\phi}{C}} \quad (30)$$

In the case of $C = \pm 2$, we have the solution: $H = H_E = 2/t$, $\phi(t) = \phi_E(t) = \pm 2\ln(Bt)$, $\xi = const$ and $V(\phi) = V_E(\phi) = 10B^2 e^{\mp 2\phi}$.

THE MODELS WITHOUT CONNECTION WITH STANDARD COSMOLOGY

Now, we consider the specific model without connection with standard cosmology. This is the case when $H_E = 0$.

With this suggestion the equations (9)–(12) are reduced to the following form

$$V(\phi) = -2H^2 + \frac{2k}{a^2} \tag{31}$$

$$\frac{1}{2}\dot{\phi}^2 = -H^2 + \frac{k}{a^2} \tag{32}$$

$$\dot{\xi} = \frac{H}{2\left(H^2 + \frac{k}{a^2}\right)} \tag{33}$$

Now, we can rewrite the equation (32) in terms of $a(t)$ from the definition of $H = \dot{a}/a$ as

$$\dot{a}^2 + \frac{1}{2}(a\dot{\phi})^2 = k \tag{34}$$

We consider the model with $k = 1$, $k > 0$ and scale factor

$$a(t) = A\sin(Bt), \tag{35}$$

with additional condition $k = A^2B^2$, where A and B are positive constants.
From equations (34), (31) and (33) we obtain

$$\phi(t) = \pm\sqrt{2}Bt, \quad V = 2B^2 \tag{36}$$

$$\xi(t) = -\frac{1}{4B^2}\ln|\cos(2Bt) + 3| + const \tag{37}$$

$$\xi(\phi) = -\frac{1}{4B^2}\ln\left|\cos(\sqrt{2}\phi) + 3\right| + const \tag{38}$$

For the open Friedmann universe ($k = -1$) we consider the phantom scalar field satisfying the equation

$$\dot{a}^2 - \frac{1}{2}(a\dot{\phi})^2 = -1 \tag{39}$$

Here, in equation (39), we changed the sign before kinetic energy $\dot{\phi}^2/2$ in equation (34) for the case of phantom field.
The next example of the exact solution is given for us by setting the scale factor as

$$a(t) = A\cosh(Bt) \tag{40}$$

with the restriction $A^2 B^2 = 1$. Thus, we have the solution

$$\phi(t) = \pm\sqrt{2}Bt, \quad V = 2B^2 \tag{41}$$

$$\xi(t) = \frac{1}{4B^2} \ln|\cosh(2Bt) - 3| + const \tag{42}$$

$$\xi(\phi) = \frac{1}{4B^2} \ln\left|\cosh\left(\sqrt{2}\phi\right) - 3\right| + const \tag{43}$$

For a spatially-flat Friedmann universe ($k = 0$) we have the general solution of equations (31)–(33) with phantom fields

$$\phi(t) = \pm\sqrt{2}\ln(a(t)) + c \tag{44}$$

$$V(t) = -2\left(\frac{\dot{a}}{a}\right)^2 \tag{45}$$

$$\xi(t) = \frac{1}{2}\int \frac{a}{\dot{a}}dt \tag{46}$$

where c is a constant of integration.

As a new example, we give the solution with the scale factor

$$a(t) = A\exp(Bt^m) \tag{47}$$

The Hubble parameter is $H = mBt^{m-1}$. From equations (44)–(46) we obtain

$$\phi(t) = \pm\sqrt{2}Bt^m + c_1, \quad c_1 = c \pm \sqrt{2}\ln A \tag{48}$$

$$V(\phi) = -2B^2m^2\left(\pm\frac{\phi - c_1}{\sqrt{2}B}\right)^{\frac{2(m-1)}{m}} \tag{49}$$

$$\xi(t) = \frac{t^{2-m}}{2(2-m)mB} + const \tag{50}$$

$$\xi(\phi) = \frac{1}{2(2-m)mB}\left(\pm\frac{\phi - c_1}{\sqrt{2}B}\right)^{\frac{2-m}{m}} + const \tag{51}$$

For the special cases $m = 1/2$ and $m = 1/3$ we have the inverse potentials $V(\phi) \propto -(\phi - c_1)^{-2}$ and $V(\phi) \propto -(\phi - c_1)^{-4}$.

GENERATION OF 4D EGB EXACT SOLUTIONS FROM A SCALE FACTOR

Now, we consider the following representation of time derivative of the GB coupling function (ansatz)

$$\dot{\xi} = \frac{Ca^3}{\dot{a}^2 + k} \qquad (52)$$

where C is an arbitrary constant.

With the representation (52) the equations (1)–(2) are reduced to the ones for the standard-like cosmology (4)–(5). Discarding the Hubble parameter via a scale factor we transform these equations to the following form

$$V(\phi) = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2} - 12C\dot{a} \qquad (53)$$

$$\frac{1}{2}\dot{\phi}^2 = -\dot{H} + \frac{k}{a^2} \qquad (54)$$

The difference from standard cosmology is the last term in the potential, therefore we can represent it as $V = V_E + V_{GB}$. The first term V_E is the potential in standard Friedman cosmology and the second term $V_{GB} = 12C\dot{a}$ corresponds to GB correction. When $C = 0$ one has $V_{GB} = 0$ and $\xi = const$.

From equation (6) with the ansatz on the coupling function (52) we obtain the Hubble parameter H_E for the model with minimal coupling

$$H_E(t) = H(t) - 2Ca(t) \tag{55}$$

The influence of non-minimal coupling of a scalar field to the GB term on the cosmological dynamics is defined by the sign of C. When $C > 0$ the non-minimal coupling rises the expansion of the universe in respect to standard Friedmann cosmological model with minimal coupling and it downwards the universe expansion in the case of $C < 0$. For $C = 0$ we have the standard Friedmann cosmology with $\xi = const$ and $H = H_E$.
Also, from relation

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} \tag{56}$$

we obtain the connection between accelerations of the universe in the cases of standard and 4D EGB cosmology

$$\frac{\ddot{a}_E}{a} = \frac{\ddot{a}}{a} - 2Ca(3H - 2Ca) \tag{57}$$

Thus, for coupling function (52), the difference between dynamics in standard and 4D EGB cosmology doesn't depend on a value of k in the framework of chosen ansatz (52).

Now, we consider the model with oscillating scale factor, besides $k \neq 0$.

$$a(t) = A\cos(Bt), \tag{58}$$

where A and B are positive constants.
From equations (52)–(54) we obtain

$$V(t) = 2B^2\tan^2(Bt) - B^2 + \frac{2k}{A^2\cos^2(Bt)} + 12ABC\sin(Bt) \tag{59}$$

$$\phi(t) = \pm\frac{\sqrt{2A^2B^2 + 2k}}{AB}\ln\left|\sec(Bt) + \tan(Bt)\right| + c_1 \tag{60}$$

$$\xi(t) = -\frac{AC}{B^3}\sin(Bt) + \left(\frac{A^2C}{B^2\sqrt{k}} + \frac{C\sqrt{k}}{B^4}\right)\arctan\left[\frac{AB}{\sqrt{k}}\sin(Bt)\right] + const \tag{61}$$

where c_1 is a constant of integration.
The Hubble parameter is

$$H(t) = -B\tan(Bt) \tag{62}$$

To simplify the formulae, we introduce two new functions

$$f(\phi) = \exp\left(\pm\frac{2AB(\phi - c_1)}{\sqrt{2A^2B^2 + 2k}}\right) \tag{63}$$

$$g(\phi) = \frac{\sqrt{f^2(\phi) + 1}}{f(\phi) + 1} \tag{64}$$

From the expression (60) we have

$$Bt = \arctan\left(\frac{f(\phi) - 1}{f(\phi) + 1}\right) \tag{65}$$

One can easily obtains the dependence $\xi = \xi(\phi)$ by means of the substitution (65) into (61)

$$\xi(\phi) = -\frac{AC}{B^3}\sin\left(\arctan\left(\frac{f(\phi)-1}{f(\phi)+1}\right)\right) + \left(\frac{A^2 C}{B^2\sqrt{k}} + \frac{C\sqrt{k}}{B^4}\right)\times$$
$$\times\arctan\left[\frac{AB}{\sqrt{k}}\sin\left(\arctan\left(\frac{f(\phi)-1}{f(\phi)+1}\right)\right)\right] + const \qquad (66)$$

Now, we can find the potential as the function of a scalar field

$$V(\phi) = \frac{1}{A^2(f(\phi)+1)^2 g(\phi)}\Big[(6\sqrt{2}A^3BC + A^2B^2)f^2(\phi) + \qquad (67)$$
$$+(4kf^2(\phi) + A^2B^2 + 4k)g(\phi) - 6A^2B^2 f(\phi)g(\phi) - 6\sqrt{2}A^3BC\Big]$$

Further, we redefined the scalar field as $\varphi = \phi - c_1 + 30$.

The potential of the similar form we have for inflation in the case of closed ($k = 1$) Friedmann universe, for open ($k = -1$) universe with $V(\varphi) > 0$ one must include the positive cosmological constant Λ in the model. For this aim, one can simply redefine the potential as $V(\varphi) \rightarrow V(\varphi) + \Lambda$.

Also, it is possible to obtain the similar potential $V_E(\varphi)$ for standard cosmology ($C = 0$, $k \neq 0$) with another model's parameters.

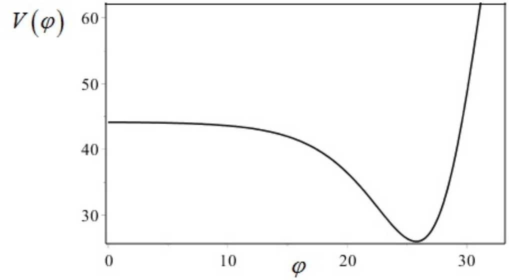

Figure: Potential $V = V(\varphi)$ with parameters $k = 30$, $A = 1.2$, $B = 1$, $C = 6$. Parameters A and C regulate the height of the potential at the point $V(\varphi = 0)$. Parameter B regulates the slope of the potential from $V(\varphi = 0)$ to V_{min}.

Now, we consider the model with Hubble parameter $H = -At$ in flat FRW universe.

$$a(t) = a_0 \exp\left(-\frac{A}{2}t^2\right) \tag{68}$$

$$\phi(t) = \sqrt{2A}t + \phi_0 \tag{69}$$

$$\xi(t) = \frac{Ca_0}{4A^2}\left[-\frac{e^{-At^2}}{t} - \sqrt{\pi A}erf\left(\sqrt{A}t\right)\right] + const \tag{70}$$

$$\xi(\phi) = \frac{Ca_0}{4A^2}\left[-\frac{\sqrt{2A}e^{-(\phi-\phi_0)^2/2}}{\phi-\phi_0} - \sqrt{\pi A}erf\left(\frac{\phi-\phi_0}{\sqrt{2}}\right)\right] + const \tag{71}$$

$$V(\phi) = \frac{3}{2}A(\phi-\phi_0)^2 + 2a_0AC(\phi-\phi_0)\exp\left(-\frac{(\phi-\phi_0)^2}{2}\right) - A + \Lambda, \quad C < 0 \tag{72}$$

The potentials for standard inflation and EGB inflation.

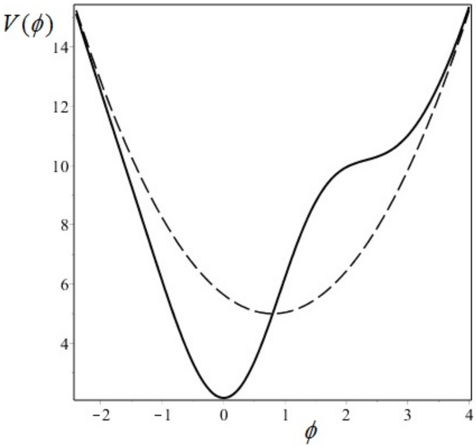

Figure: Potential $V = V(\phi)$ with parameter $C = 0$ and $C = -5$.

E-FOLDS NUMBER

In the case of a spatially-flat Friedmann universe we can estimate the influence of the GB term on the dynamics by means of difference between e-folds numbers for 4D EGB and standard inflation

$$\Delta_N = N - N_E = \int_{t_i}^{t_e} (H - H_E) dt = 2 \int_{t_i}^{t_e} \dot{\xi} H^2 dt,$$

where t_i and t_i are the times of the beginning and the end of inflation.
For $\xi = const$ we obtain $\Delta_N = 0$, in the case of $H_E = 0$ we have $\dot{\xi} = 1/2H$ and

$$\Delta_N = N = \int_{t_i}^{t_e} H dt$$

For models with coupling function (52) we have

$$\Delta_N = N - N_E = 2C \int_{t_i}^{t_e} a(t) dt,$$

thus, difference between e-folds numbers depends on the dynamics and the value of constant C.

CONCLUSION: PROGRESS AND PROBLEMS

PROGRESS:

- **FINDING THE ESTIMATION OF NONMINIMAL COUPLING IN EGB COSMOLOGY ON THE DYNAMICS OF THE UNIVERSE**

- **THE EFFECTIVE TECHNIQUE OF EXACT SOLUTIONS OF SCALAR FIELD DYNAMICAL EQUATIONS**

PROBLEMS:

- **THE CALCULATION OF EXACT PARAMETERS OF COSMOLOGICAL PERTURBATIONS**

- **THE GENERALIZATION OF THE CONSIDERED METHOD FOR ANOTHER MODELS OF GRAVITY**